University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 4

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1. A chain of mass M and length L falls onto a table. Initially, the chain is hanging so that its lower end just touches the table. The chain is falling in gravity, so the velocity of a link that is falling is given by $v = gt$. The distance that the chain has fallen is given by $x = gt^2/2$. These two facts tell us how much of the chain is on the table at a given time. The density of the chain is M/L, so the mass of chain on the table is just Mx/L:

$$M(t) = \frac{1}{2}\frac{M}{L}gt^2$$

The rate at which the mass is falling on the table is just

$$\frac{d}{dt}M(t) = \frac{M}{L}gt = \frac{M}{L}v(t)$$

At time t, the free elements of chain are moving with speed $v(t)$. This is the velocity they have when they hit the table. The total rate at which momentum is being transferred is

$$\frac{dp}{dt} = v(t)\frac{d}{dt}M(t) = \frac{M}{L}v^2(t)$$

Writing this in terms of $M(t)$, the mass on the table at time t, we get the following:

$$\frac{dp}{dt} = 2M(t)g$$

The rate of change of momentum should be familiar to you from Newton's second law which states

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

Thus the table must be exerting this force on the chain to slow it down. Remember also that the table exerts a normal force on the chain which is equal to the force of gravity

$$F_N = M(t)g$$

Thus the total force that the table exerts on the chain is three times the weight of the chain on the table:

$$F(t) = F_N(t) + 2M(t)g = 3M(t)g$$

2. The airspeed of a plane is $v = 1000$ m/sec. The engines take in 80 kg of air per second and mix it with 30 kg of fuel. The mixture is expelled after it ignites, and it is moving at a velocity of 3000 m/sec relative to the plane. We can calculate the thrust of this engine by calculating the rate of change of momentum. The fuel is ejected at a rate of 30 kg/sec, and it is given a velocity of 3000 m/sec relative to the plane. It started at rest with respect to the plane, so it need to be given the full velocity. The rate of change of momentum this corresponds to is

$$\frac{dp}{dt} = \frac{dm}{dt}v = 30 \times 3000 = 90{,}000 \text{ kg} - \text{m/sec}^2$$

The air also contributes to the momentum. It is expelled at a rate of 80 kg/sec. Its velocity is already 1000 m/sec relative to the plane, so it only needs to gain 2000 m/sec of velocity in the engine. The rate of change of momentum that this corresponds to is

$$\frac{dp}{dt} = \frac{dm}{dt}v = 80 \times 2000 = 160{,}000 \text{ kg} - \text{m/sec}^2$$

The total rate of momentum transferred to the exhaust by the plane's engine is thus

$$\frac{dp}{dt} = 250{,}000 \text{ kg} - \text{m/sec}^2$$

This rate of momentum transfer is equal to the thrust of the engine:

$$F_{\text{thrust}} = 250{,}000 \text{ N}$$

3. K&K problem 3.13 This problem concerns the total force that a ski lift must exert to lift skiers to the top of a hill. There will be two parts to the force. The first is just the force necessary to oppose the force of gravity on the skiers. The second is the force required to accelerate the skier at the bottom from rest to the speed of the lift. The rope is 100 meters long, and it is pulled at 1.5 meters per second. On average, one skier uses the tow rope every five seconds. This means the tow rope travels $5 \times 1.5 = 7.5$ meters between skiers, so $100/7.5 = 13\frac{1}{3}$ skiers are on the rope on average. Each skier weighs 70 kg, so the average total weight of the skiers who are on the rope is 933 kg. The component of the force of gravity that must be offset by the rope is determined by the angle of the slope, which is 20°. The component of the acceleration of gravity that is directed down the slope is just $g \sin 20° = 0.342g$. Therefore the force that the tow rope must exert to offset that component of gravity is $933 \times g \times 0.342 = 3128$ N. In addition, when a skier grabs the rope, he must be accelerated to the speed of the rope, 1.5 m/sec. The change in momentum for the skier is $1.5 \times 70 = 105$ kg-m/sec. This change in momentum must be provided by the motor once every five seconds, which is how often skiers use the lift. On average, this force is 105/5=21 N. Therefore the total force that the lift must provide is, on average, 3128+21=3149 N.

4. A two stage rocket carries a payload of mass m. The total mass of the rocket is Nm, and the mass of the second stage and payload is nm. In each stage, the mass of the fuel is a fraction (1-r) of the total, so the mass of the casing is a fraction r of the total mass. The first stage has a mass $(N - n)m$,which is just the total minus the mass of the second stage.

(a.) Since gravity can be ignored, the equation for rocket motion derived in class reduces to

$$v(t) - v_0 = V \ln \frac{M_0}{M(t)} .$$

To determine the velocity gain v from the first burn, we need only to compute the mass of the rocket at the end of the burn. The initial mass is Nm, while the mass of fuel burned by the first stage is $(Nm - nm)$, the mass of the first stage, multiplied by $(1-r)$. The difference $m(n+r(N-n))$ of these two masses is the residual mass after the first burn. So the first burn velocity gain is

$$v = V \ln \frac{N}{n + r(N-n)} = V \ln \frac{N}{Nr + n(1-r)} .$$

(b.) The method for this part is the same as for part (a.) because the first equation guarantees that the velocity gain of a rocket is independent of its initial velocity. Here the initial mass is the full mass of the second stage, nm. The final mass is nm minus the mass of fuel consumed in the second burn, which is $(1-r)(nm-m)$. This yields $m(1 + r(n-1))$ for the final mass, and a second burn velocity gain of

$$u = V \ln \frac{n}{1 + r(n-1)} = V \ln \frac{n}{nr + (1-r)} .$$

(c.) Here we optimize n with all other parameters fixed. We wish to maximize $v + u$. As V is fixed, we choose equivalently to minimize $Q = \ln (V/(v+u))$ in order to simplify the algebra. From (a.) and (b.) we have

$$Q = \frac{Nr + n(1-r)}{N} \frac{nr + (1-r)}{n} .$$

Carrying out the division,

$$Q = (r + \frac{n}{N}(1-r))(r + \frac{1}{n}(1-r)) .$$

Multiplying,

$$Q = r^2 + \frac{(1-r)^2}{N} + r(1-r)(\frac{n}{N} + \frac{1}{n}) .$$

Only the last term depends on n:

$$\frac{d}{dn}\left(\frac{n}{N} + \frac{1}{n}\right) = 0, \quad n = \sqrt{N} .$$

(d.) For this value of n, the velocity gains from the first and second burns are equal:

$$v + u = 2u = 2V \ln \frac{\sqrt{N}}{1 + r(\sqrt{N} - 1)} .$$

(e.) A single stage rocket has the same values of N, r, and V. The initial mass is Nm, as in part (a.), and the final mass is $m + r(Nm - m)$, in analogy to part (b.) with N substituted for n. The final velocity is

$$v = V \ln \frac{N}{Nr + (1-r)} \ .$$

(f.) We want the final velocity of the payload to be $v = 10$ km/sec, and we have a rocket with exhaust velocity $V = 2.5$ km/sec and $r = 0.1$. First let's see if this can be done with a single stage rocket. Plugging into the result from part (e.), we see that

$$10 = 2.5 \ln \frac{N}{0.9 + 0.1N}$$

We try to solve for the necessary N

$$\begin{aligned} e^4 &= \frac{N}{0.9 + 0.1N} \\ 5.46N + 49.1 &= N \\ N &= -11.0 \ . \end{aligned}$$

This answer doesn't make any sense, which means that a single stage rocket can't do the job. Let's now look at the optimal two stage rocket, using the result from part (d.):

$$10 = 5 \ln \frac{N}{0.1N + 0.9\sqrt{N}}$$

Again we try to solve for N:

$$\begin{aligned} e^2 &= \frac{N}{0.1N + 0.9\sqrt{N}} \\ 0.739N + 6.65\sqrt{N} &= N \\ 0.261\sqrt{N} &= 6.65 \\ N &= 650 \ . \end{aligned}$$

This rocket indeed can be built.

5. A boat of mass M and length L is floating at rest. A man of mass m is sitting at the stern. He stands up, walks to the bow and sits down again.

(a.) There is no force from the water, therefore the net force on the system is zero. The momentum of the system is conserved, and the center of mass remains at the same velocity, in this case zero. Centering the boat at $x = 0$, we can calculate the center of mass

$$X_{CM} = \frac{-(L/2)m}{M+m}$$

After the man is sitting at the bow, the center of the boat will be at some position x, which means that the man will be at a position $x + (L/2)$. However, the center of mass will be in the same place.

$$\begin{aligned} \frac{Mx + m(x + (L/2))}{M+m} &= \frac{-(L/2)m}{M+m} \\ x &= -\frac{mL}{M+m} \ . \end{aligned}$$

The boat has moved from its initial position.

(b.) This time, the water exerts a viscous force $F = -kv$ on the boat. We can show that the boat will always return to its original position. Newton's second law gives the following equation. We want to use the total mass of the boat plus the man, because we don't want the man accelerating relative to the boat

$$(m+M)\dot{v} = -kv \Rightarrow \int_{v_0}^{v(t)} \frac{dv}{v} = -\frac{k}{M+m} \int_{t_0}^{t} dt$$

This gives

$$v(t) = v_0 \exp\left(-\frac{k}{M+m}(t - t_0)\right)$$

The distance traveled in this interval is just the integral of the velocity

$$x(t) = \frac{(M+m)v_0}{k}\left(1 - e^{-\frac{k}{M+m}(t-t_0)}\right)$$

Now we just need to find the initial velocity of the boat. When the man starts moving, say he applies an impulse Δp. This is the same impulse that the boat must receive, but in the opposite direction. Thus, the velocity of the man is $u = \Delta p/m$ and the velocity of the boat is $v = -\Delta p/M$. This means that the velocity of the man relative to the boat is $u - v = (m+M)\Delta p/Mm$. The man is now walking at constant speed relative to the boat. We plug in the initial velocity of the boat

$v = -\Delta p/M$ to the solution of the differential equation and we find the velocity of the boat

$$v(t) = -\frac{\Delta p}{M} \exp\left(-\frac{k}{M+m}(t-t_0)\right)$$

At time $\tau = L/(u-v) = LMm/(M+m)\Delta p$, the man has reached the other end of the boat. The velocity of the boat is

$$v(\tau) = -\frac{\Delta p}{M} \exp\left(-\frac{k}{M+m}\tau\right)$$

and it has traveled a distance

$$x(\tau) = -\frac{(M+m)\Delta p}{kM}\left(1 - \exp\left(-\frac{k}{M+m}\tau\right)\right)$$

He again applies an impulse, but this time it is $-\Delta p$. This gives the boat a change in velocity of $+\Delta p/M$. The total velocity of the boat is now

$$v(\tau) = \frac{\Delta p}{M}\left(1 - \exp\left(-\frac{k}{M+m}\tau\right)\right)$$

Using this as the initial velocity, we again solve the differential equation

$$v(t) = \frac{\Delta p}{M}\left(1 - e^{-\frac{k}{M+m}\tau}\right) e^{-\frac{k}{M+m}t}$$

This is correct for all $t > \tau$. We now calculate the total distance traveled in the second part of the trip. We take the final time to be $t = \infty$.

$$x(\infty) = \frac{(M+m)\Delta p}{kM}\left(1 - \exp\left(-\frac{k}{M+m}\tau\right)\right)$$

This is exactly the opposite of the distance traveled in the first part. Thus the boat will eventually return to its starting point.

(b′.) Here is a quick, elegant way to prove the result of part (b.). It deserves full grading credit. We do not mention only this method because, as seen above, the problem is amenable to solution by systematic calculation as well as brilliant insight.

Consider the impulse applied by the force F_{ext} of the water on the boat. To specify the impulse, which is the time integral of F_{ext}, we must specify the time interval. We choose the interval from $t = 0-$, just before any motion starts, to $t = \infty$, at which time all motion must have stopped due to effects of viscosity. At both of those times the total momentum of the boat+man system, whose rate of change is controlled by F_{ext}, is zero. Therefore the impulse in question, which is equal to the difference $P(\infty) - P(0-)$ of the boat+man system, must vanish.

The same impulse can also be written as

$$\begin{aligned} 0 &= \int_{0-}^{\infty} F_{\text{ext}} dt \\ &= -k \int_{0-}^{\infty} \frac{dx}{dt} dt = -k \int_{0-}^{\infty} dx \\ &= -k(x(\infty) - x(0-)) , \end{aligned}$$

where x is the position of the boat. This proves that the boat returns to its original position.

(c.) The result of part (b.) says that any viscous force, no matter how small, results in the boat returning to its original location. The result of part (a.) says that when there is no viscous force, the boat moves some distance. Mathematically, the difference between the two results is due to the order in which the limits are taken. In part (a.), the first thing done is to take the limit as $k \to 0$, no viscous force. Then the limit as $t \to \infty$ is taken. If we look at the result of part (b.), we first take the limit as $t \to \infty$, then we consider what happens when there is no viscous force. This is an instance in which we cannot reverse the order of taking limits. Denoting the results from part (a.) and (b.) by capital letters, we see that

$$\lim_{t\to\infty} \lim_{k\to 0} A \neq \lim_{k\to 0} \lim_{t\to\infty} B$$

So much for the reason why, mathematically, the results of (a.) and (b.) are not the same. Physically, they are not in conflict. As the coefficient k approaches zero in part (b.), the speed with which the boat ultimately migrates back to its original position approaches zero also. This cannot be distinguished by physical measurement from the limiting case (a).

6. The Great Pyramid at Gizeh is h=150 m high and has a square base of side $s = 230$ m. It has a density $\rho = 2.5$ g/cc.

(a.) If all the stone is initially at ground level, it must be raised to its position in the pyramid. The work required to this is

$$W = Mgh_{cm} = \int \rho g z \, dV$$

The volume element is the area of the square at a height z times dz, the differential of height. The square has side s at $z = 0$ and side 0 at $z = h$. The width of the square decreases linearly with height, so the width and area at height z is given by

$$w(z) = s\left(1 - \frac{z}{h}\right) \qquad A(z) = s^2\left(1 - \frac{z}{h}\right)^2$$

The volume element dV is given by $dV = A(z)\, dz$. We can now perform the integral. Expanding the polynomial in z

$$W = \rho g s^2 \int_0^h \left(z - 2\frac{z^2}{h} + \frac{z^3}{h^2}\right) dz$$

This is a simple integral to perform:

$$W = \rho g s^2 h^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{12}\rho g s^2 h^2$$

Plugging in the values for these constants, we get the amount of work required to erect the pyramid

$$W = 2.43 \times 10^{12} \text{ Joules}$$

(b.) The slaves employed in building this pyramid consumed 1500 Calories per day, which is 6.3×10^6 joules per day. With 100,000 slaves working for 20 years, this is 730 million slave-days of work to build the pyramid. The total energy the slaves spent is thus 4.6×10^{15} joules. The efficiency thus implied is low, $\epsilon = 5.3 \times 10^{-4}$. This does not necessarily reflect a low intrinsic efficiency, since the slaves undoubtedly expended most of their energy on activities other than lifting the stone blocks to their final position.

7. A force $\mathbf{f}$(t) has magnitude F at $t = 0$, magnitude 0 at $t = T$, and it decreases linearly with time. The direction remains the same. The magnitude of the force is thus

$$f(t) = F\left(1 - \frac{t}{T}\right)$$

The force acts on a particle of mass m initially at rest. The kinetic energy at $t = T$ is just the integral

$$K = \int_0^T F\left(1 - \frac{t}{T}\right) dx = \int_0^T F\left(1 - \frac{t}{T}\right) v\, dt$$

We can find v by applying Newton's second law, but once we have it, we don't need to do the integral because we know that $K = mv^2/2$

$$f(t) = m\frac{dv}{dt} = F\left(1 - \frac{t}{T}\right)$$

We can just directly integrate both sides with respect to t, with limits $t = 0$ and $t = T$

$$v(T) = \frac{F}{2m}T$$

We now have the answer

$$K = \frac{1}{8}\frac{F^2T^2}{m}$$

8. Instantaneously after the collision of the bullet and block, after the bullet has come to rest but before the frictional force on the block has had time to slow it down more than an infintesimal amount, we can apply momentum conservation to the bullet-block collision. At that time the total momentum of the block+bullet system is $(M + m)v_0'$, where v_0' is the velocity of the block+bullet system immediately after the collision. Momentum conservation requires that momentum to be equal to the initial momentum mv of the bullet. Thus

$$v_0' = \frac{mv}{M + m} \ .$$

After the collision, the normal force on the block+bullet system from the table is $(M + m)g$, giving rise to a frictional force

$$\mu N = \mu(M + m)g$$

on the sliding block+bullet system. This causes a constant acceleration μg of that system opposite to its velocity.

Take $t = 0$ at the time of collision. Afterward, the block+bullet system's velocity in the horizontal direction will be $v'(t) = v_0' - \mu g t$. It will continue sliding until $v'(t) = 0$, at which point the frictional force will disappear and it will remain at rest. Solving, the time at which the block-bullet system stops is

$$t = v_0'/(\mu g) \ .$$

The distance traveled in that time is

$$x = v_0' t - \frac{1}{2}\mu g t^2 = \frac{1}{2} v_0' t = \frac{(v_0')^2}{2\mu g} \ .$$

Plugging in the already deduced value for v_0', this distance is

$$x = \left(\frac{m}{M+m}\right)^2 \frac{v^2}{2\mu g} \ .$$